AN OPTIMAL WEGNER ESTIMATE AND ITS APPLICATION TO THE GLOBAL CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR RANDOM SCHRÖDINGER OPERATORS

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Abstract

We prove that the integrated density of states (IDS) of random Schrödinger operators with Anderson-type potentials on $L^2(\mathbb{R}^d)$, for $d \geq 1$, is locally Hölder continuous at all energies with the same Hölder exponent $0 < \alpha \leq 1$ as the conditional probability measure for the single-site random variable. As a special case, we prove that if the probability distribution is absolutely continuous with respect to Lebesgue measure with a bounded density, then the IDS is Lipschitz continuous at all energies. The single-site potential $u \in L_0^\infty(\mathbb{R}^d)$ must be nonnegative and compactly-supported. The unperturbed Hamiltonian must be periodic and satisfy a unique continuation principle. We also prove analogous continuity results for the IDS of random Anderson-type perturbations of the Landau Hamiltonian in two-dimensions. All of these results follow from a new Wegner estimate for local random Hamiltonians with rather general probability measures.

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1 Introduction and Main Results

In this paper, we combine approaches of [4] and [6] to prove, as a special case, the Lipschitz continuity of the integrated density of states (IDS) for random Schrödinger operators $H_\omega = H_0 + V_\omega$, on $L^2(\mathbb{R}^d)$, for $d \geq 1$, provided the conditional probability distribution for the random variable at a single-site has a density in $L_0^\infty(\mathbb{R})$. In previous papers [6, 7], we proved global Hölder continuity, for any order strictly less than one, of the IDS under the same hypotheses on the single-site probability measure, and, in [18], there was an improvement up to a logarithmic factor (see below). It has long been expected that if the probability measure of a single-site random variable has a bounded density with compact support, then the IDS should be locally Lipschitz continuous at all energies. This is known to be true if the single-site potential satisfies a simple covering condition [4, 5]. This result is a special case of the continuity bound proved in this paper. We prove that if the conditional probability measure is Hölder continuous of order $0 < \alpha \leq 1$, then the IDS is Hölder continuous of order α at all energies. Hence, the IDS has at least the same continuity property as the conditional probability measure. These results follow from a Wegner estimate valid for a very general class of probability measures. We refer to [6] for an introduction to the problem and discussion of previous results.

The family of Schrödinger operators $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^d)$, is constructed from a deterministic, periodic, background operator $H_0 = (-i\nabla - A_0)^2 + V_0$. We assume that this operator is self-adjoint with operator core $C_0^\infty(\mathbb{R}^d)$, and that $H_0 \geq -M_0 > -\infty$, for some finite constant M_0. We consider an Anderson-type potential V_ω constructed from the nonzero single-site potential $u \geq 0$ as

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j). \tag{1.1}$$

We assume very little on the random variables $\{\omega_j \mid j \in \mathbb{Z}^d\}$ except that they form a bounded, real-valued process over $\mathbb{Z}^d$ with probability space $(I\!P, \Omega)$. We remark that the results of this paper also apply to the random operators describing acoustic and electromagnetic waves in randomly perturbed media, and we refer the reader to [11, 13, 14].

We need to define local versions of the Hamiltonians and potentials associated with bounded regions in $\mathbb{R}^d$. By $\Lambda_l(x)$, we mean the open cube of side length l centered at $x \in \mathbb{R}^d$. For $\Lambda \subset \mathbb{R}^d$, we denote the lattice points in Λ

by $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$. For a cube Λ, we take H_0^Λ and H_ω^Λ to be the restrictions of H_0 and H_ω, respectively, to the cube Λ, with periodic boundary conditions on the boundary $\partial\Lambda$ of Λ. We denote by $E_0^\Lambda(\cdot)$ and $E_\Lambda(\cdot)$ the spectral families for H_0^Λ and H_ω^Λ, respectively. Furthermore, for $\Lambda \subset \mathbb{R}^d$, let χ_Λ be the characteristic function for Λ. The local potential V_Λ is defined by

$$V_\Lambda(x) = V_\omega(x)\chi_\Lambda(x), \tag{1.2}$$

and we assume this can be written as

$$V_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} \omega_j u(x-j). \tag{1.3}$$

For example, if the support of u is contained in a single unit cube, formula (1.3) holds. We refer to the discussion in [6] when the support of u is compact, but not necessarily contained inside one cube. In this case, V_Λ can be written as in (1.3) plus a boundary term of order $|\partial\Lambda|$ and hence it does not contribute to the large $|\Lambda|$ limit. Hence, we may assume (1.3) without any loss of generality. We will also use the local potential obtained from (1.3) by setting all the random variables to one, that is,

$$\tilde{V}_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} u(x-j). \tag{1.4}$$

We will always make the following four assumptions:

(H1). The background operator $H_0 = (-i\nabla - A_0)^2 + V_0$ is a lower semi-bounded, $\mathbb{Z}^d$-periodic Schrödinger operator with a real-valued, $\mathbb{Z}^d$-periodic, potential V_0, and a $\mathbb{Z}^d$-periodic vector potential A_0. We assume that V_0 and A_0 are sufficiently regular so that H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

(H2). The periodic operator H_0 has the unique continuation property, that is, for any $E \in \mathbb{R}$ and for any function $\phi \in H^2_{loc}(\mathbb{R}^d)$, if $(H_0 - E)\phi = 0$, and if ϕ vanishes on an open set, then $\phi \equiv 0$.

(H3). The nonzero, nonnegative, compactly-supported, single-site potential $u \in L_0^\infty(\mathbb{R}^d)$, with $\|u\|_\infty \leq 1$, and it is strictly positive on a nonempty open set.

(H4). The nonconstant random coupling constants $\{\omega_j \mid j \in \mathbb{Z}^d\}$ take values in $[m_0, M_0]$ and form a real-valued, bounded process $\mathbb{Z}^d$ with probability space $(I\!\!P, \Omega)$.

Note that the condition on $\|u\|_\infty$ in (H3) can always be obtained by rescaling the random variables.

Our main technical result under hypotheses (H1)–(H4) is an optimal Wegner estimate expressed in Theorem 1.3. This upper bound (1.10) is optimal with respect to the volume dependence and the dependence on the distribution of the random variables. This implies the continuity results for the IDS expressed in Theorems 1.1 – 1.2. In order to describe the dependence on the probability measure $I\!\!P$, we let μ_j denote the conditional probability measure for the random variable ω_j at site $j \in \mathbb{Z}^d$, conditioned on all the random variables $(\omega_k)_{k\neq j}$, that is

$$\mu_j([E, E+\epsilon]) = \mathbb{P}\{\omega_j \in [E, E+\epsilon] \,|\, (\omega_k)_{k\neq j}\} \tag{1.5}$$

The Wegner estimate and continuity results for the IDS are expressed in terms of the following quantity:

$$s(\epsilon) \equiv \sup_{j\in\mathbb{Z}^d} I\!\!E \left\{ \sup_{E\in\mathbb{R}} \ \mu_j([E, E+\epsilon]) \right\}. \tag{1.6}$$

Clearly, if the $(\omega_j)_{j\in\mathbb{Z}^d}$ are independent, μ_j is just the probability measure of the random variable ω_j. If, in addition, the random variables ω_j are identically distributed, then all the μ_j are the same, which we write as μ_0, and $(I\!\!P, \Omega)$ is the usual product probability space.

Our results on the Wegner estimate and the IDS are of greatest interest if the function $s(\epsilon)$, defined in (1.6), satisfies $s(\epsilon) \to 0$, when $\epsilon \to 0^+$. In applications to continuity of the IDS or Anderson localization, the rate of vanishing of $s(\epsilon)$, as $\epsilon \to 0^+$, is essential. If, for example, in the case of independent and identically distributed (*iid*) random variables, the measure μ_j is concentrated on a discrete set, our results do not provide this control.

We make two comments on hypotheses (H1)–(H4). First, concerning the unique continuation property, it is well known that H_0 has the UCP if A_0 and V_0 are sufficiently regular; e.g. in dimension $d \geq 3$, $V_0 \in L^{d/2}_{loc}(\mathbb{R}^d)$, $A_0 \in L^d_{loc}(\mathbb{R}^d)$ and $\nabla A_0 \in L^{d/2}_{loc}(\mathbb{R}^d)$ are sufficient to ensure that H_0 has the UCP (see e.g. [34] and references therein). It also follows that the Landau

Hamiltonian (1.7) has the UCP. Second, the boundedness of the random variables is not essential. The results can be generalized to a class of unbounded random variables.

We define the IDS $N(E)$ for H_ω using the counting function for H_ω^Λ. Let $N_\Lambda(E)$ be the number of eigenvalues of H_ω^Λ, with periodic boundary conditions, less than or equal to E. This function depends on the realization ω. The integrated density of states (IDS) is defined by

$$N(E) = \lim_{|\Lambda| \to \infty} \frac{N_\Lambda(E)}{|\Lambda|},$$

when this limit exists. As assumptions (H1)-(H4) do not guarantee the existence of this limit, we will always assume the following.

(H5). The IDS $N(E)$ exists almost surely for the random family of operators considered here.

Because $N(E)$ is a monotonic function, we assume that $N(E)$ has been defined to be right continuous, and it has at most a countable number of discontinuities. For example, if the family H_ω is an ergodic family of random Schrödinger operators, it is known that this limit exists and is independent of the realization ω almost surely (cf. [3, 20, 26]). Furthermore, it is known that the IDS is independent of the boundary conditions taken on the finite volumes Λ, cf. [12, 20, 25]. Our main new result on the IDS is the following theorem.

Theorem 1.1 *Assume that the family of random Schrödinger operators* $H_\omega = H_0 + V_\omega$ *on* $L^2(\mathbb{R}^d)$*, for* $d \geq 1$*, satisfies hypotheses (H1)-(H5). Then, for any* $I \subset \mathbb{R}$ *compact, there exists* $C_I > 0$ *such that for any* $E \in I$ *and for any* $\epsilon \in (0, 1]$*, one has*

$$0 \leq N(E + \epsilon) - N(E) \leq C_I \, s(\epsilon),$$

where $s(\epsilon)$ *is defined in* (1.6).

As pointed out above, in order to apply this result to Anderson localization or to the continuity of the IDS, we need to impose conditions on the probability measure $I\!P$ so that the function $\epsilon \mapsto s(\epsilon)$ vanishes as $\epsilon = 0^+$. A case of particular interest is when the random variables $(\omega_j)_{j \in \mathbb{Z}^d}$ satisfy

not only (H4) but are also *iid* with a common probability measure μ_0 that is locally Hölder continuous of order $0 < \alpha \leq 1$. That is, if for any interval $[a, b] \subset \text{ supp } \mu_0$, we have $\mu_0([a, b]) \leq C_0 |b - a|^\alpha$, for some finite, positive constant $C_0 > 0$ (locally bounded). The function $s(\epsilon)$ in (1.6) then satisfies $s(\epsilon) \leq C_{\mu_0} \epsilon^\alpha$. Theorem 1.1 states that in this case the IDS $N(E)$ for the random family H_ω is locally Hölder continuous with uniform Hölder exponent α. That is, for any bounded, closed interval $I \subset \mathbb{R}$, there is a finite positive constant $0 \leq C_I < \infty$, so that for any $E, E' \in I$, the IDS satisfies

$$|N(E') - N(E)| \leq C_I |E' - E|^\alpha.$$

If $\alpha = 1$, then the IDS is locally Lipschitz continuous on $\mathbb{R}$. This condition on the probability measure μ_0 is stronger than just the absolute continuity of the probability measure as it implies that it admits a nonnegative, bounded, compactly-supported density h_0. Note that, in the *iid* case, the existence of the IDS is well known, hence, assumption (H5) can be dropped. We have the following simple, but important, corollary.

Corollary 1.1 *Suppose the random family satisfies (H1)-(H3) and the random variables $(\omega_j)_{j \in \mathbb{Z}^d}$ are iid and the common probability measure μ_0 is locally Lipschitz continuous and compactly supported. Then the IDS $N(E)$ is locally uniformly Lipschitz continuous and the density of states $\rho(E)$ exists as a locally bounded function.*

We remark that Corollary 1.1 follows from the new analysis in section 2 and the spectral averaging result of [4] that is valid for a compactly-supported, Lipschitz continuous probability measure μ_0. In particular, the new spectral averaging result presented in Theorem 3.1 is not needed for this case.

We next consider the IDS for random Anderson-type perturbations of Landau Hamiltonians. The unperturbed operator $H_L(B)$ on $L^2(\mathbb{R}^2)$ has the form

$$H_L(B) = (-i\nabla - A)^2, \quad \text{where} \quad A(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \tag{1.7}$$

where $B > 0$ is the magnetic field strength. The spectrum is pure point and consists of an increasing sequence of degenerate, isolated eigenvalues $\{E_j(B) = (2j+1)B \mid j = 0, 1, \ldots, \}$ of infinite multiplicity. The unperturbed Hamiltonian $H_L(B)$ satisfies the unique continuation principle as stated in

(H2). The IDS for this model is a piecewise constant, monotone increasing function (cf. the example in [25]). The perturbed family of operators is

$$H_\omega = H_L(B) + V_\omega, \tag{1.8}$$

where V_ω is the Anderson-type random perturbation given in (1.1). It is known that $N(E)$ is locally Lipschitz continuous in the following sense. Given an $N > 0$, there is a $B_N > 0$ so that for $B > B_N$, the IDS $N(E)$ is Lipschitz continuous on $(0, 2(N+1)B)\backslash\{E_j(B) \mid j = 0, 1, \ldots, N\}$ [5, 31]. Under some additional conditions, Wang [32] also proved that $N(E)$ is smooth outside of a given Landau level for sufficiently large magnetic field strength. There has been some discussion as to the behavior of the IDS at the Landau energies $E_j(B)$. If the single-site potential u in (1.1) has support including the unit cube $\Lambda_1(0)$ and satisfies $u|\Lambda_1(0) > \epsilon\chi_{\Lambda_1(0)} > 0$, for some $\epsilon > 0$, then the IDS is locally Lipschitz continuous at all energies [5]. The following theorem improves [6] and [7]. Note that the result holds for any nonzero flux.

Theorem 1.2 *Let H_ω be the perturbed Landau Hamiltonian (1.7)-(1.8) with magnetic field $B \neq 0$. Suppose that this family satisfies (H3)–(H5). Then, for any $I \subset \mathbb{R}$ compact, there exists $C_I > 0$ such that for any $E \in I$ and for any $\epsilon \in (0, 1]$, one has*

$$0 \leq N(E+\epsilon) - N(E) \leq C_I\, s(\epsilon),$$

where $s(\epsilon)$ is defined in (1.6).

Of course the remarks following Theorem 1.1, in particular Corollary 1.1, hold for the randomly perturbed Landau Hamiltonian.

Both main results, Theorems 1.1 and 1.2, are proved by establishing a Wegner estimate for the local Hamiltonians H_Λ and using the identity

$$|N(E+\epsilon) - N(E)| \leq \liminf_{|\Lambda| \to \infty} I\!E \left\{ \frac{1}{|\Lambda|} Tr E_\Lambda([E, E+\epsilon]) \right\}, \tag{1.9}$$

for ϵ small enough. We prove a new Wegner estimate in this paper that holds for general probability measures. The Wegner estimate is also essential in many proofs of Anderson localization using the method of multiscale analysis.

Theorem 1.3 *Assume that the family of random Schrödinger operators $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^d)$ satisfies hypotheses (H1)-(H4). Then, there exists a locally uniform constant $C_W > 0$ such that for any $E_0 \in \mathbb{R}$, and $\epsilon \in (0,1]$, the local Hamiltonians H_Λ satisfy the following Wegner estimate*

$$\begin{aligned} I\!P\{ dist(\sigma(H_\Lambda), E_0) < \epsilon\} &\leq I\!E\{Tr E_\Lambda([E_0 - \epsilon, E_0 + \epsilon])\} \\ &\leq C_W s(2\epsilon)|\Lambda|, \end{aligned} \tag{1.10}$$

where $s(\epsilon)$ is defined in (1.6). A similar estimate holds for randomly perturbed Landau Hamiltonians.

As an application of our results to a situation involving correlated random variables, we consider the family of nonsign definite single-site potentials introduced by Veselić [30]. Let $\Gamma \subset \mathbb{Z}^d$ be a finite set of vectors indexed by $k = 0, \ldots, |\Gamma| < \infty$ (we refer to $k \in \Gamma$). We consider a family of bounded, real-valued variables α_j, for $j \in \Gamma$. We assume that $\sum_{j \neq 0} |\alpha_j| < |\alpha_0|$. This condition guarantees the invertibility of a certain Toeplitz matrix constructed from the α_j. Let w be a single-site potential as in (H3) and define a new single-site potential u by

$$u(x) = \sum_{j \in \Gamma} \alpha_j w(x - j). \tag{1.11}$$

Since the coefficients are not required to have fixed sign, the potential u is not sign definite. We now construct an Anderson-type random potential with *iid* random variables ω_j as in (1.1). Upon substituting the definition of u in (1.11) into (1.1), we can write the potential as

$$V_\eta(x) = \sum_{j \in \mathbb{Z}^d} \eta_j u(x - j), \tag{1.12}$$

where the new family of random variables $\eta_j = \sum_{k \in \Gamma} \alpha_{j-k} \omega_k$ is no longer independent. They form a correlated process with finite-range determined by Γ. It is easy to compute the conditional probability measure μ_j for the random variables η_j from the distribution for the variables ω_k. In particular, if the single-site probability distribution μ_0 for ω_0 has a density, then so does the conditional probability measure μ_j. Theorem 1.1 applies to this case and as a result the IDS is Lipschitz continuous at all energies. Veselić required that u have a large support satisfying $u \geq C_0 \chi_{\Lambda_1(0)}$, but our results apply for u as in (H3).

There are very few results on the Wegner estimate for general processes on $\mathbb{Z}^d$. In the *iid* case, Stollmann [28] considered a general compactly-supported probability measure μ_0 and, using a completely different method, proved a Wegner estimate of the form (1.10) but with a volume factor of $|\Lambda|^2$, rather than $|\Lambda|$ as in Theorem 1.3. Stollmann's result can be used to prove Anderson localization for Hölder continuous probability measures using the multiscale analysis but, because of the $|\Lambda|^2$-factor, cannot be used to study the IDS. More recently, Hundertmark, Killip, Nakamura, Stollmann, and Veselić [18] obtained a bound of the form $s(\epsilon)[\log(1/\epsilon)]^d|\Lambda|$, improving Stollmann's bound to the correct volume factor, but under the strong assumption that $u \geq c_0\chi_{\Lambda_1(0)}$, the characteristic function of the unit cube $\Lambda_1(0)$. In Theorem 1.3, this covering condition is no longer necessary. The result in [18] follows from a new exponentially decreasing bound, in the index n, on the n^{th} singular value of the difference of two semigroups generated by Hamiltonians H_1 and H_2 for which the perturbation $H_1 - H_2$ has compact support. This estimate is used to improve the estimate on the spectral shift function obtained in [10]. Using these estimates, the authors improve the Hölder continuity of the IDS in the Hölder continuous situation studied in [6] obtaining $\epsilon[\log(1/\epsilon)]^d$, in place of ϵ^p, for any $0 < p < 1$.

The contents of this paper are as follows. We prove Theorem 1.3, which implies Theorem 1.1, in section 2, assuming a key spectral averaging result. We prove this new spectral averaging result for general, compactly-supported probability measures in section 3. We prove the corresponding result, Theorem1.2, for randomly perturbed Landau Hamiltonians, in section 4. In the first appendix, section 6, we prove some necessary trace estimates.

Applications of Theorem 1.3 to pointwise bounds on the expectation of the spectral shift function are presented in [8].

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2 Proof of Theorem 1.1

We now prove Theorem 1.1 via (1.9) by proving a Wegner estimate (1.10). We always assume that u is nonzero so that V_ω is nonzero. Recall that by

the operators H_0^Λ and H_ω^Λ, we mean the operators H_0 and H_ω restricted to the cube Λ with periodic boundary conditions. We will often write H_Λ for H_ω^Λ. Their spectral families are denoted by $E_0^\Lambda(\cdot)$ and $E_\Lambda(\cdot)$, respectively. In [6], we proved

Theorem 2.1 *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Γ-periodic, nonnegative function. Suppose that $V > 0$ on some open set. Consider a bounded interval $I \subset \mathbb{R}$. Then, there exists a finite constant $C(I, V) > 0$ such that, for any $\Lambda \subset \mathbb{R}^d$ cube with integral edges (i.e. vertices in $\mathbb{Z}^d$), one has,*

$$\mathbf{E}_0^\Lambda(I) V_\Lambda \mathbf{E}_0^\Lambda(I) \geq C(I, V) \mathbf{E}_0^\Lambda(I)$$

where V_Λ is the restriction of V to Λ.

This clearly yields that there exists a constant $C(\tilde{\Delta}, u) > 0$, independent of Λ, so that

$$E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \ \geq C(\tilde{\Delta}, u) E_0^\Lambda(\tilde{\Delta}). \tag{2.1}$$

For a fixed, but arbitrary, $E_0 \in \mathbb{R}$, let $E_0 \in \Delta \subset \tilde{\Delta}$ be two closed, bounded intervals centered on E_0, and let $d_\Delta \equiv$ dist $(\Delta, \tilde{\Delta}^c)$. We will always assume that $d_\Delta > 0$.

Preparatory to the proof of Theorem 1.1, we note that hypothesis (H3) implies the following. There exists a finite constant $D_0 \equiv D_0(u, d) > 0$, depending only on the single-site potential u, and the dimension $d \geq 1$, so that for all $\Lambda \subset \mathbb{R}^d$,

$$0 \ \leq \ \tilde{V}_\Lambda^2 \ \leq \ D_0(u, d) \tilde{V}_\Lambda, \tag{2.2}$$

where $\tilde{V}_\Lambda$ is defined in (1.4). We will use this in the proof.

Proof of Theorem 1.1
1. Recalling that $E_\Lambda(\Delta)$ is a trace class operator, we need to estimate

$$I\!\!E\{Tr E_\Lambda(\Delta)\}. \tag{2.3}$$

We begin with a decomposition relative to the spectral projectors $E_0^\Lambda(\cdot)$ for the operator H_0^Λ. We write

$$Tr E_\Lambda(\Delta) = Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) + Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c), \tag{2.4}$$

where the intervals $\Delta \subset \tilde{\Delta}$ satisfy $|\Delta| < 1$ and $d_\Delta > 0$. If $\tilde{\Delta}$, and consequently Δ, lies in a spectral gap of H_0, then only the second term on the

right in (2.4) contributes and the result follows from (2.16). Hence, we only need to consider the case when Δ does not lie in a spectral gap of H_0.
2. The term involving $\tilde{\Delta}^c$ is estimated as follows. Since $E_\Lambda(\Delta)$ is trace class, let $\{\phi_m^\Lambda\}$ be the set of normalized eigenfunctions in its range. We expand the trace in these eigenfunctions and obtain

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) = \sum_m \langle \phi_m^\Lambda, E_0^\Lambda(\tilde{\Delta}^c)\phi_m^\Lambda \rangle. \tag{2.5}$$

From the eigenfunction equation $(H_\omega^\Lambda - E_m)\phi_m^\Lambda = 0$, we easily obtain

$$-(H_0^\Lambda - E_m)^{-1} E_0^\Lambda(\tilde{\Delta}^c) V_\Lambda \phi_m^\Lambda = E_0^\Lambda(\tilde{\Delta}^c)\phi_m^\Lambda.$$

Substituting this into the right side of (2.5), and resumming to obtain a trace, we find

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) = \sum_m \langle \phi_m^\Lambda, \left(V_\Lambda \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda - E_m)^2} V_\Lambda \right) \phi_m^\Lambda \rangle. \tag{2.6}$$

We next want to replace the energy $E_m \in \Delta$ in the resolvent in (2.6) by a fixed number, say $-M$, assuming $H_0^\Lambda > -M > -\infty$. To do this, we define an operator K by

$$K \equiv \left(\frac{H_0^\Lambda + M}{H_0^\Lambda - E_m} \right)^2 E_0^\Lambda(\tilde{\Delta}^c), \tag{2.7}$$

and note that K is bounded, independent of m, by

$$\|K\| \leq K_0 \equiv \left[1 + \frac{2(M + \Delta_+)}{d_\Delta} + \frac{(M + \Delta_+)^2}{d_\Delta^2} \right],$$

where $\Delta = [\Delta_-, \Delta_+]$. Now, for any $\psi \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \left\langle \psi, \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda - E_m)^2}\psi \right\rangle &\leq \left\langle \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda + M)}\psi, K \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda + M)}\psi \right\rangle \\ &\leq K_0 \left\langle \psi, \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda + M)^2}\psi \right\rangle \\ &\leq K_0 \left\langle \psi, \frac{1}{(H_0^\Lambda + M)^2}\psi \right\rangle, \end{aligned} \tag{2.8}$$

since $E_0^\Lambda(\tilde{\Delta}^c) \leq 1$. We use the bound (2.8) on the right in (2.6) and expand the potential. To facilitate this, let $\chi \geq 0$ be a function of compact support slightly larger than the support of u, and so that $\chi u = u$. We set $\chi_j(x) = \chi(x-j)$, for $j \in \mathbb{Z}^d$. Returning to (2.6), we obtain the bound

$$\begin{aligned} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) &\leq K_0 \, TrE_\Lambda(\Delta)\left(V_\Lambda \frac{1}{(H_0^\Lambda + M)^2} V_\Lambda\right) \\ &\leq K_0 \sum_{i,j\in\tilde{\Lambda}} |\omega_i\omega_j| \left| Tr\left[u_j E_\Lambda(\Delta) u_i \cdot \left(\chi_i \frac{1}{(H_0^\Lambda + M)^2}\chi_j\right)\right]\right| \\ &\leq K_0 \sum_{i,j\in\tilde{\Lambda}} \left| Tr\left[u_j E_\Lambda(\Delta) u_i \cdot \left(\chi_i \frac{1}{(H_0^\Lambda + M)^2}\chi_j\right)\right]\right|. \end{aligned} \tag{2.9}$$

3. We divide the double sum in (2.9) into two terms: For fixed $i \in \tilde{\Lambda}$, one sum is over $j \in \tilde{\Lambda}$ for which $\chi_i\chi_j = 0$, and in the second sum is over the remaining $j \in \tilde{\Lambda}$ so that $\chi_i\chi_j \neq 0$. For the first sum, we note that the operator $K_{ij} \equiv \chi_i(H_0^\Lambda + M)^{-2}\chi_j$ in (2.6) is trace class for $d = 1, 2, 3$. Furthermore, we prove in Lemma 6.1 that the operator K_{ij} is trace class in all dimensions when $\chi_i\chi_j = 0$, and the trace norm $\|K_{ij}\|_1$ decays exponentially in $\|i-j\|$ as

$$\|K_{ij}\|_1 = \|\chi_i(H_0^\Lambda + M)^{-2}\chi_j\|_1 \leq C_0 e^{-c_0\|i-j\|}, \tag{2.10}$$

for positive constants $C_0, c_0 > 0$ depending on M. To control the second sum in (2.9), we define, for each $i \in \tilde{\Lambda}$, an index set $\mathcal{J}_i = \{j \in \tilde{\Lambda} \mid \chi_i\chi_j \neq 0\}$. We note that $|\mathcal{J}_i|$ depends only on u, and is independent of i and Λ. We define an operator $\tilde{K}_\Lambda$ by

$$\tilde{K}_\Lambda \equiv \sum_{i\in\tilde{\Lambda};j\in\mathcal{J}_i} \chi_j K_{ij} \chi_i. \tag{2.11}$$

In Lemma 6.1, we prove that for any $m > 0$, and $\sigma_j > 0$, for $j = 0, 1, \ldots, m$,

$$\begin{aligned} \left|\sum_{i\in\tilde{\Lambda};j\in\mathcal{J}_i} Tr\ u_j E_\Lambda(\Delta) u_i \cdot K_{ij}\right| &\leq \left(\sum_{j=1}^m \frac{\sigma_j}{2^j\sigma_1\cdots\sigma_{j-1}}\right) TrE_\Lambda(\Delta) \\ &\quad + \left(\frac{1}{2^m\sigma_1\cdots\sigma_m}\right) Tr\ E_\Lambda(\Delta)\cdot\tilde{K}_\Lambda^{2^m}, \end{aligned} \tag{2.12}$$

and that if $m+2 > \log d/\log 2$, the operator $\tilde{K}_\Lambda^{2^m}$ is trace class and $\|\tilde{K}_\Lambda^{2^m}\|_1 \leq C(\chi, m, d)|\Lambda|$. We next choose the σ_j in Lemma 6.1 so that the term involving $TrE_\Lambda(\Delta)$ in (2.12) can be moved to the left in (2.4). Since the coefficient in (2.9) is K_0, we choose $\sigma_1 = K_0^{-1}$, and successively $\sigma_j = K_0^{-2^{j-1}}$. Then, the coefficient in (2.12) is $(1-2^{-m})K_0^{-1}$.

4. We now return to estimating the right side of (2.9). We have seen that in the disjoint support case, the operator $K_{ij} \in \mathcal{I}_1$, and in the nondisjoint support case, we must work with $\tilde{K}_\Lambda^{2^m} \in \mathcal{I}_1$, for m large enough. We first show how to control the expectation of the trace on the far right of (2.12). For simplicity, we write $n = 2^m$ and recall the sets $\mathcal{J}_{j_k}$ defined in the proof of Lemma 6.1. First, we write this trace as

$$Tr\ E_\Lambda(\Delta) \cdot \tilde{K}_\Lambda^n = \sum_{i \in \tilde{\Lambda}; j \in \mathcal{J}_{j_{n-1}}} Tr\ u_{j_n} E_\Lambda(\Delta) u_i \cdot \tilde{K}(n)_{ij_n}. \tag{2.13}$$

As in Lemma 6.1, the operator $K\tilde{(n)}_{ij}$ is trace class. The canonical representation of $K\tilde{(n)}_{ij}$ (where we write j for j_n) is

$$\tilde{K}(n)_{ij} = \sum_l \mu_l^{(ij)} |\phi_l^{(ij)}\rangle\langle\psi_l^{(ij)}|$$

where $(\phi_l^{(ij)})_l$, $(\psi_l^{(ij)})_l$ are orthonormal families and $\displaystyle\sum_l |\mu_l^{(ij)}| < +\infty$.

Inserting this into the trace (2.13), we obtain

$$\begin{aligned} Tr\ E_\Lambda(\Delta) \cdot \tilde{K}_\Lambda^n &\leq \sum_{i \in \tilde{\Lambda}; j \in \mathcal{J}_{j_{n-1}}} \sum_l \mu_l^{(ij)} \langle \psi_l^{(ij)}, u_j E_\Lambda(\Delta) u_i \phi_l^{(ij)} \rangle \\ &\leq \sum_{i \in \tilde{\Lambda}; j \in \mathcal{J}_{j_{n-1}}} \sum_l \mu_l^{(ij)} \left\{ \langle \psi_l^{(ij)}, u_j E_\Lambda(\Delta) u_j \psi_l^{(ij)} \rangle + \right. \\ &\qquad \left. \langle \phi_l^{(ij)}, u_i E_\Lambda(\Delta) u_i \phi_l^{(ij)} \rangle \right\}. \end{aligned} \tag{2.14}$$

We will prove in section 3 below that the expectation of the matrix elements in (2.14) satisfy the following bound

$$I\!E\{\langle \psi_l^{(ij)}, u_j E_\Lambda(\Delta) u_j \psi_l^{(ij)} \rangle\} \leq 8s(|\Delta|), \tag{2.15}$$

where $s(\epsilon)$ is defined in (1.6). It follows from (2.6)-(2.14) and the bound (2.15) that

$$\begin{aligned} I\!E\{Tr\ E_\Lambda(\Delta)\cdot \tilde{K}_\Lambda^n\} &\leq \sum_{i\in\tilde{\Lambda};j\in\mathcal{J}_{j_{n-1}}} C(\chi)s(|\Delta|)\ \|\tilde{K}(n)_{ij}\|_1 \\ &\leq C(\chi,m)s(|\Delta|)|\Lambda|. \end{aligned} \tag{2.16}$$

We use the same technique for the disjoint support terms for which the exponential decay in the trace norm (2.10) controls the double sum to give one factor of $|\Lambda|$. Returning to (2.9), we obtain

$$I\!E(TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)) \leq K_0C(u,m)s(|\Delta|)|\Lambda|,$$

plus a term involving $TrE_\Lambda(\Delta)$ with a coefficient less than one from (2.12) that is moved to the left in (2.4).

5. As for the first term on the right in (2.4), we use the fundamental assumption (2.1). As in [4], we will use the spectral projector $E_0(\tilde{\Delta})$ of H_0^Λ in order to control the trace. We have

$$\begin{aligned} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) &\leq \frac{1}{C(\tilde{\Delta},u)}\left\{TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\right\} \\ &\leq \frac{1}{C(\tilde{\Delta},u)}\Big\{TrE_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \\ &\qquad -TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\Big\}. \end{aligned} \tag{2.17}$$

We estimate the second term on the right in (2.17). Using the Hölder inequality for trace norms, we have, for any $\kappa_0 > 0$,

$$\begin{aligned} &|TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \\ &\leq \|E_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)\|_2\ \|\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta)\|_2 \\ &\leq \frac{1}{2\kappa_0}TrE_0^\Lambda(\tilde{\Delta}^c)E_\Lambda(\Delta) + \frac{\kappa_0}{2}TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda^2E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta). \end{aligned} \tag{2.18}$$

We next estimate the second term on the right in (2.18). Let D_0 be the constant in (2.2) so that $\tilde{V}_\Lambda^2 \leq D_0\tilde{V}_\Lambda$. Using this, we find that for any $\kappa_1 > 0$,

$$\begin{aligned} &TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda^2E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta) \\ &\leq D_0\|E_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda\|_2\ \|E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta)\|_2 \\ &\leq \frac{D_0\kappa_1}{2}TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda^2E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta) + \frac{D_0}{2\kappa_1}TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}). \end{aligned}$$

We choose $\kappa_1 = 1/D_0 > 0$ so that $(1 - D_0\kappa_1/2) = 1/2$. Consequently, we obtain

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta) \ \le\ D_0^2 TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}). \tag{2.19}$$

Inserting this into (2.18), we find

$$|TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \ \le\ \frac{1}{2\kappa_0}TrE_0^\Lambda(\tilde{\Delta}^c)E_\Lambda(\Delta) + \frac{\kappa_0 D_0^2}{2}TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}). \tag{2.20}$$

As a consequence of (2.20), we obtain for the first term on the right in (2.4),

$$\begin{aligned}
&\left(1 - \frac{\kappa_0 D_0^2}{2C(\tilde{\Delta}, u)}\right) TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \\
&\le\ \frac{1}{C(\tilde{\Delta}, u)}|TrE_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| + \frac{1}{2\kappa_0 C(\tilde{\Delta}, u)}\ TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c).
\end{aligned}$$

We choose $\kappa_0 = C(\tilde{\Delta}, u)/D_0^2$ so that we have

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \le \frac{2}{C(\tilde{\Delta}, u)}|TrE_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| + \frac{D_0^2}{C(\tilde{\Delta}, u)^2}\ TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c). \tag{2.21}$$

As for the first term on the right in (2.21), we use Hölder's inequality and write

$$\begin{aligned}
&|TrE_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \\
&\le\ \|E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta)\|_2\ \|E_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\|_2 \\
&\le\ \frac{1}{2\sigma}\|E_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta)\|_2^2 + \frac{\sigma}{2}\|E_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\|_2^2 \\
&\le\ \frac{1}{2\sigma}TrE_0^\Lambda(\tilde{\Delta})E_\Lambda(\Delta) + \frac{\sigma}{2}TrE_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda E_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}),
\end{aligned} \tag{2.22}$$

for any constant $\sigma > 0$. In light of the coefficient in (2.21), we choose $\sigma = 2/C(\tilde{\Delta}, u)$ and obtain from (2.21) and (2.22),

$$\begin{aligned}
TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \ \le\ & \frac{4}{C(\tilde{\Delta}, u)^2}TrE_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda E_\Lambda(\Delta)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \\
& + \frac{2D_0^2}{C(\tilde{\Delta}, u)^2}\ TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c).
\end{aligned} \tag{2.23}$$

The second term on the right in (2.23) is bounded above as in (2.14) and (2.16).

6. We estimate the first term on the right in the last line of (2.23). Let $f_\Delta \in C_0^\infty(\mathbb{R})$ be a smooth, compactly-supported, nonnegative function $0 \leq f \leq 1$, with $f_\Delta \chi_\Delta = \chi_\Delta$, where χ_Δ is the characteristic function on Δ. Note that we can take $|\text{supp } f| \sim 1$ so that the derivatives of f are order one. By positivity, we have the bound

$$\begin{aligned} & Tr E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \\ & \quad = \; Tr E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \\ & \quad \leq \; Tr E_\Lambda(\Delta) \tilde{V}_\Lambda f_\Delta(H_0^\Lambda) \tilde{V}_\Lambda E_\Lambda(\Delta). \end{aligned} \tag{2.24}$$

Recall that χ_j is a compactly-supported function so that $u_j \chi_j = u_j$. Upon expanding the potential $\tilde{V}_\Lambda$, the term on the right in (2.24) is

$$\sum_{j,k \in \tilde{\Lambda}} Tr\; u_k E_\Lambda(\Delta) u_j \cdot \chi_j f_\Delta(H_0^\Lambda) \chi_k. \tag{2.25}$$

The operator $\chi_j f_\Delta(H_0^\Lambda)\chi_k$ is a nonrandom, trace class operator. As with the operator K_{ij} in (2.9), it admits a canonical representation

$$\chi_j f_\Delta(H_0^\Lambda) \chi_k = \sum_l \lambda_l^{(jk)} |\phi_l^{(jk)}\rangle \langle \psi_l^{(jk)}|, \tag{2.26}$$

for orthonormal functions $\phi_l^{(jk)}$ and $\psi_l^{(jk)}$. This operator also satisfies a decay estimate of the type

$$\|\chi_j f_\Delta(H_0^\Lambda) \chi_k\|_1 \leq C_N(f)(1 + \|k - j\|^2)^{-N}, \tag{2.27}$$

for any $N \in \mathbb{N}$ and a finite positive constant depending on $\|f^{(j)}\|$ independent of $|\Delta|$. This can be proved using the Helffer-Sjöstrand formula, see, for example, [16]. Expanding the trace in (2.25) as in (2.14), we can bound (2.25) from above by

$$\begin{aligned} Tr E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \; &= \; \sum_l \sum_{j,k \in \tilde{\Lambda}} \lambda_m^{(jk)} \; \langle \psi_l^{(jk)}, u_j E_\Lambda(\Delta) u_k \phi_l^{(jk)} \rangle \\ &\leq \; \sum_l \sum_{j,k \in \tilde{\Lambda}} \lambda_l^{(jk)} \{ \langle \psi_l^{(jk)}, u_j E_\Lambda(\Delta) u_j \psi_l^{(jk)} \rangle \\ & \qquad + \langle \phi_l^{(jk)}, u_k E_\Lambda(\Delta) u_k \phi_l^{(jk)} \rangle \}. \end{aligned} \tag{2.28}$$

As in (2.16), the expectation of the matrix elements of the projector $E_\Lambda(\Delta)$ of the type occurring in (2.28) are bounded above as

$$\mathbb{E}\{\langle \xi, u_l E_\Lambda(\Delta) u_l \xi \rangle\} \leq 8 s(|\Delta|), \tag{2.29}$$

where $\|\xi\| = 1$, and $s(\epsilon)$ is defined in (1.6). Given this bound, and the decay bound (2.27), we obtain

$$\begin{aligned} \mathbb{E}\{Tr E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\} &\leq 2 \left(\sum_{j,k \in \tilde{\Lambda}} \|\chi_j f_\Delta(H_0^\Lambda) \chi_k\|_1 \right) C_0(u) s(|\Delta|) \\ &\leq C_1(u) s(|\Delta|) |\Lambda|. \end{aligned} \tag{2.30}$$

This estimate, together with estimate (2.16) and inequality (2.21), prove that

$$\mathbb{E}\{Tr E_\Lambda(\Delta)\} \leq C_2(u) s(|\Delta|) |\Lambda|. \tag{2.31}$$

this proves the Wegner estimate of Theorem 1.3. The results on the IDS in Theorem 1.1 now follows from this Wegner estimate, the additional Hölder continuity hypothesis, and the fact that

$$s(|\Delta|) \leq C_3 |\Delta|^\alpha, \tag{2.32}$$

for some locally uniform constant $C_3 > 0$. □

3 Spectral Averaging for General Probability Measures

We now turn to the proof of (2.15) and (2.29) for general probability measures. As noted after Corollary 1.1 in section 1, a local Lipschitz condition on the random variables implies the existence of a bounded density $h_0 \in L^\infty_{loc}(\mathbb{R})$ with compact support. Hence, this case can be treated by the spectral averaging method of [4, 9, 23]. For the general case, we now present a new one-parameter averaging method.

We consider the one-parameter family of operators $H_\Lambda(\omega_j) = H^\Lambda_{j\perp} + \omega_j u_j$, where $H^\Lambda_{j\perp}$ is H_Λ with $\omega_j = 0$. Let $E_0 \in \mathbb{R}$ be fixed and arbitrary. We consider an interval $\Delta_\epsilon = [E_0, E_0 + \epsilon]$, for some fixed $0 < \epsilon < \infty$. A simple

use of the spectral theorem for a self-adjoint operator H with spectral family $E_H(\cdot)$ shows that

$$
\begin{aligned}
&\int_{\Delta_\epsilon} dE \ \langle \phi, \Im (H - E - i\epsilon)^{-1} \phi \rangle \\
&= \ \langle \phi, \left[\tan^{-1} \left(\frac{E_0 + \epsilon - H}{\epsilon} \right) - \tan^{-1} \left(\frac{E_0 - H}{\epsilon} \right) \right] \phi \rangle \\
&\geq \ (\tan^{-1} 1) \langle \phi, E_H(\Delta_\epsilon) \phi \rangle = (\pi/4) \langle \phi, E_H(\Delta_\epsilon) \phi \rangle. \qquad (3.1)
\end{aligned}
$$

Applying this to the matrix element in (2.29), we obtain

$$
\langle \phi, u_j E_\Lambda(\Delta_\epsilon) u_j \phi \rangle \leq (\frac{4}{\pi}) \int_{\Delta_\epsilon} dE \ \Im \ \langle u_j \phi, \frac{1}{H^\Lambda_{j^\perp} + \omega_j u_j - E - i\epsilon} u_j \phi \rangle. \qquad (3.2)
$$

Our goal is to evaluate the expectation of the matrix element in (3.2) with respect to the random variable ω_j. To this end, we prove a new spectral averaging result that is a discretized version of previous spectral averaging results.

Theorem 3.1 *Let A and B be two self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and suppose that B is bounded and non negative. Then, for any $\phi \in \mathcal{H}$, we have the bound*

$$
\sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \langle B\phi, \frac{1}{(A + (n+y)B)^2 + 1} B\phi \rangle \leq \pi \|B\| (1 + \|B\|) \|\phi\|^2. \qquad (3.3)
$$

The proof of Theorem 3.1 uses two technical tools: the following Lemma 3.1, the simple proof of which is left to the reader, and Theorem 3.2 that utilizes a basic result from the theory of maximally dissipative operators, that we briefly recall below.

For $\kappa \in \mathbb{R}$, and $b > 0$, we define the function

$$
\ell(\kappa; b) \equiv \sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \frac{b}{(y + n + \kappa)^2 + b^2}. \qquad (3.4)
$$

Lemma 3.1 *For $b > 0$, the function $\kappa \mapsto \ell(\kappa; b)$ is $\mathbb{Z}$-periodic and satisfies the bound*

$$
\sup_{\kappa \in \mathbb{R}} \ell(\kappa; b) \leq \pi \left(1 + \frac{1}{b} \right). \qquad (3.5)
$$

Next, we recall a main result in the theory of maximally dissipative operators (cf. [24, 29]). A closed operator A is maximally dissipative if $\Im A \geq 0$ and A has no proper dissipative extension.

Proposition 3.1 *Suppose A is a maximally dissipative operator on a separable Hilbert space $\mathcal{H}$. Then, there exists a Hilbert space $\tilde{\mathcal{H}}$, containing $\mathcal{H}$ as a subspace, an orthogonal projection $P : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$, and a self-adjoint dilation L so that for $z \in \mathbb{C}$ with $\Im z < 0$,*

$$(A - z)^{-1} = P(L - z)^{-1} P^*. \tag{3.6}$$

Note that the signs of the imaginary parts in the denominator of the left side of (3.6) are the same. Consequently, the result is valid for an operator A if $-A$ is maximally dissipative provided $\Im z > 0$. Also note that under the conditions in Proposition 3.1, we have $\Im (A - z)^{-1} \leq 0$.

Lemma 3.1 and Proposition 3.1 allows us to prove the following theorem.

Theorem 3.2 *Let A be a maximally dissipative operator and let $B \geq 0$ be a bounded, nonnegative self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Fix $\lambda > 0$. Then, for any $\phi \in \mathcal{H}$, we have the bound*

$$-\sum_{n \in \mathbb{Z}} \sup_{y \in [0,1]} \Im \langle B^{1/2} \phi, \frac{1}{A + (n+y)B + i\lambda B} B^{1/2} \phi \rangle \leq \pi \left(1 + \frac{1}{\lambda} \right) \|\phi\|^2. \tag{3.7}$$

Proof: Let $\delta > 0$ be a small parameter and set $B_\delta \equiv B + \delta > \delta$, since $B \geq 0$. As B_δ is bounded and invertible, we can write

$$\langle B_\delta^{1/2} \phi, \frac{1}{A + (n+y)B_\delta + i\lambda B_\delta} B_\delta^{1/2} \phi \rangle = \langle \phi, \frac{1}{B_\delta^{-1/2} A B_\delta^{-1/2} + (n+y) + i\lambda} \phi \rangle. \tag{3.8}$$

Since $B \geq 0$ and bounded, and A is maximally dissipative, so is $B_\delta^{-1/2} A B_\delta^{-1/2}$. Let P and L be the orthogonal projector and self-adjoint dilation associated with $B_\delta^{-1/2} A B_\delta^{-1/2}$ as in Proposition 3.1. Let μ_L^ψ be the spectral measure for L and the vector ψ. We can write the matrix element in (3.8) as

$$\langle P^* \phi, \frac{1}{L + (n+y) + i\lambda} P^* \phi \rangle = \int_{\mathbb{R}} d\mu_L^{P^* \phi}(s) \frac{1}{s + (n+y) + i\lambda}. \tag{3.9}$$

Inserting (3.9) into (3.8), taking the imaginary part, summing over $n \in \mathbb{N}$, taking the supremum over $y \in [0,1]$, and using Fubini's Theorem to intervert summation and integration, we obtain

$$\begin{aligned} & -\sum_{n\in\mathbb{Z}} \sup_{y\in[0,1]} \Im\langle B_\delta^{1/2}\phi, \frac{1}{A+(n+y)B_\delta + i\lambda B_\delta} B_\delta^{1/2}\phi\rangle \\ & \quad \leq \int_{\mathbb{R}} d\mu_L^{P^*\phi}(s) \left(\sup_{\kappa\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \sup_{y\in[0,1]} \frac{\lambda}{(y+n+\kappa)^2+\lambda} \right). \end{aligned} \tag{3.10}$$

By (3.5), the right side of (3.10) is bounded above by $\pi(1+\lambda^{-1})\|\phi\|^2$ and we obtain the bound (3.7) with B_δ in place of B. Now, $B_\delta \to B$ in norm, and the resolvent $(A+(n+y)B_\delta + i\lambda B_\delta)^{-1}$ converges to $(A+(n+y)B+i\lambda B)^{-1}$, uniformly in y. It follows that each term of the series in (3.7), with B_δ in place of B, converges to the corresponding term with $\delta = 0$, and the result follows by Fubini's Theorem. $\square$.

Proof of Theorem 3.1: We derive Theorem 3.1 from Theorem 3.2. Pick $0 < \lambda < \|B\|^{-1}$. We write the matrix element on the left in (3.3) as

$$\begin{aligned} \langle B\phi, \frac{1}{(A+(n+y)B)^2+1} B\phi\rangle &= -\Im\langle B\phi, \frac{1}{A+(n+y)B+i} B\phi\rangle \\ &= -\Im\langle B\phi, \frac{1}{[A+(1-\lambda B)i]+(n+y)B+i\lambda B} B\phi\rangle. \end{aligned}$$

The operator $A+(1-\lambda B)i$ is maximally dissipative as A is self-adjoint and $1-\lambda B \geq 1-\lambda\|B\| > 0$ (see e.g. Lemma B.1 in [1]). We apply Theorem 3.2 with ϕ replaced with $B^{1/2}\phi$ and thus obtain

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \sup_{y\in[0,1]} \langle B\phi, \frac{1}{(A+(n+y)B)^2+1} B\phi\rangle &\leq \pi(1+\lambda^{-1})\|B^{1/2}\phi\|^2 \\ &\leq \pi\|B\|(1+\lambda^{-1})\|\phi\|^2. \end{aligned}$$

Letting λ tend to $\|B\|^{-1}$, this immediately yields (3.3). This completes the proof of Theorem 3.1. $\square$.

We can now prove the necessary estimate on the expectation of the integral in (3.2) for general probability measures.

Proposition 3.2 *Let μ_j denote the probability measure of the random variable ω_j conditioned on all the random variables $(\omega_k)_{k\neq j}$ and let $s(\epsilon)$ be as defined in (1.6). Assume (H4) is satisfied. For any $\epsilon > 0$, let $\Delta_\epsilon \subset \mathbb{R}$ be an interval with $|\Delta_\epsilon| = \epsilon$. We have the following bound on the expectation of the energy integral appearing in (3.2):*

$$I\!E \left\{ \int_{\Delta_\epsilon} dE \int_{\mathbb{R}} d\mu_j(\omega_j) \; \Im \langle \phi, u_j \left(\frac{1}{H^\Lambda_{j^\perp} + \omega_j u_j - E - i\epsilon} \right) u_j \phi \rangle \right\} \leq 2\pi s(\epsilon) \|\phi\|^2. \tag{3.11}$$

Proof: The imaginary part of the matrix element in (3.11) is

$$\langle u_j\phi, \frac{\epsilon}{(H^\Lambda_{j^\perp} - E + \omega_j u_j)^2 + \epsilon^2} u_j\phi\rangle = \frac{1}{\epsilon} \langle u_j\phi, \frac{1}{\epsilon^{-2}(H^\Lambda_{j^\perp} - E + \omega_j u_j)^2 + 1} u_j\phi\rangle. \tag{3.12}$$

To apply Theorem 3.1, we choose $B = u_j$ and define a self-adjoint operator $A \equiv \epsilon^{-1}(H^\Lambda_{j^\perp} - E)$ so the matrix element in (3.12) may be written as

$$\langle B\phi, \frac{1}{(A + \epsilon^{-1}\omega_j B)^2 + 1} B\phi\rangle. \tag{3.13}$$

We divide the integration over ω_j into a sum over intervals $[n\epsilon, (n+1)\epsilon]$, and change variables letting $\omega_j/\epsilon = n + y$, so that $y \in [0, 1]$. We then obtain

$$\begin{aligned} & I\!E \left\{ \int_{\mathbb{R}} d\mu_j(\omega_j) \langle B\phi, \frac{1}{(A + \epsilon^{-1}\omega_j B)^2 + 1} B\phi\rangle \right\} \\ &= I\!E \left\{ \sum_n \int_{n\epsilon}^{(n+1)\epsilon} d\mu_j(\omega_j) \langle B\phi, \frac{1}{(A + (n+y)B)^2 + 1} B\phi\rangle \right\} \\ &\leq I\!E \left\{ \left(\sup_{m\in\mathbb{Z}} \mu_j([m\epsilon, (m+1)\epsilon]) \right) \sum_n \sup_{y\in[0,1]} \langle B\phi, \frac{1}{(A + (n+y)B)^2 + 1} B\phi\rangle \right\} \end{aligned} \tag{3.14}$$

We apply Theorem 3.1 to the last line in (3.14) and obtain

$$\begin{aligned} I\!E \left\{ \int_{\mathbb{R}} d\mu_j(\omega_j) \langle B\phi, \frac{1}{(A + \epsilon^{-1}\omega_j B)^2 + 1} B\phi\rangle \right\} &\leq 2\pi \|\phi\|^2 \; I\!E\{[\sup_m \mu_j([m\epsilon, (m+1)\epsilon])]\} \\ &\leq 2\pi \|\phi\|^2 s(\epsilon), \end{aligned} \tag{3.15}$$

since $\|B\| = \|u_j\| \leq 1$. This provides a bound for the average over ω_j of (3.12). Integrating in energy over Δ_ϵ, and recalling the factor of ϵ^{-1} in (3.12), we obtain the estimate (3.11). $\Box$

We combine (3.1) with (3.11) to obtain

$$I\!\!E\{\langle \phi, u_j E_\Lambda(\Delta) u_j \phi \rangle\} \leq 8s(\epsilon)\|\phi\|^2, \tag{3.16}$$

which is (2.15) and (2.29).

4 The Integrated Density of States for Random Landau Hamiltonians

The method of proof in section 2 can be adapted to treat randomly perturbed Landau Hamiltonians. The unperturbed Landau Hamiltonian $H_L(B)$ on $L^2(\mathbb{R}^2)$ is described in (1.7), and the perturbed operator H_ω in (1.8). The random potential V_ω is Anderson-type as in (1.1). A quantitative version of the unique continuation principle for infinite-volume Landau Hamiltonians, analogous to (2.1), was proved in [7]. We note that this result holds independent of the flux.

Theorem 4.1 *Let $H_L(B)$ be the Landau Hamiltonian in (1.7) and let Π_n be the projector onto the infinite-dimensional eigenspace for $H_L(B)$ corresponding to the eigenvalue $E_n(B)$. Let $u \geq 0$, the single-site potential, be a nonnegative, compactly-supported function with $u \in L^\infty(\mathbb{R}^2)$, and satisfying $u > u_0 > 0$ on some nonempty open set, for some constant $u_0 > 0$. We define the potential $\tilde{V}$ by*

$$\tilde{V}(x) \equiv \sum_{j \in \mathbb{Z}^2} u(x - j).$$

Then, there exists a finite constant $0 < C_n(B, u) < \infty$, so that

$$\Pi_n \tilde{V} \Pi_n \ \geq \ C_n(B, u) \Pi_n. \tag{4.1}$$

This infinite-volume result was used in [7] to prove the local Hölder continuity of the IDS, and could be used here to improve the result to local Hölder continuity with exponent $0 < \alpha \leq 1$. However, it is easier to pursue a purely local result and also obtain a Wegner estimate. Motivated by transport questions for random Landau Hamiltonians (1.8), Germinet, Klein, and

Schenker [17] used the result (4.1) to prove a purely local version of the quantitative unique continuation principle. This allowed them to prove a Wegner estimate for Landau Hamiltonians at any energy, including the Landau levels. With this result, we show how to use the method of proof in section 2 to obtain an improved Wegner estimate and, consequently, an improved continuity estimate on the IDS.

As in [17], given a magnetic field strength $B > 0$, we define a number $K_B \equiv \min\{k \in \mathbb{N} \mid k \geq \sqrt{B/4\pi}\}$, and a length scale $L_B \equiv K_B\sqrt{B/4\pi}$. Corresponding to L_B we define a set of length scales $\mathbb{N}_B = L_B\mathbb{N}$. For squares of side length L_BN, the flux is an even integer. The local, unperturbed Landau Hamiltonians $H^0_{\Lambda_L}(B)$ are defined on squares $\Lambda_L(0)$, with $L \in \mathbb{N}_B$, with periodic boundary conditions consistent with the magnetic translations. The spectrum of these local operators is discrete and consists of finite multiplicity eigenvalues at the Landau levels $E_n(B)$. We denote by $\Pi_{n,L}$ the finite rank projection onto the eigenspace corresponding to the n^{th} Landau level $E_n(B)$. The local random Hamiltonians associated with squares $\Lambda_L(0)$ are defined by $H_\Lambda(B) = H^0_{\Lambda_L}(B) + V_\Lambda$, where

$$V_\Lambda(x) = \sum_{j \in \tilde{\Lambda}_{L-\delta_u}(0)} \omega_j u(x-j),$$

and $\operatorname{supp} u \subset \Lambda_{\delta_u}(0)$. We obtain local Hamiltonians for squares $\Lambda_L(x)$ by conjugation with the magnetic translation group operators considered as maps from $L^2(\Lambda_L(0)) \to L^2(\Lambda_L(x))$. We always consider $B > 0$ fixed.

Theorem 4.2 *[17] There exists a finite, positive constant $C(n,u) > 0$, independent of $L \in \mathbb{N}_B$ large enough, so that*

$$\Pi_{n,L}\tilde{V}_{\Lambda_L}\Pi_{n,L} \geq C(n,u)\Pi_{n,L}. \tag{4.2}$$

We now sketch the proof of the following Wegner estimate from which the main Theorem 1.2 follows. The local random Hamiltonians $H_\Lambda(B)$ are defined above with $L \in \mathbb{N}_B$ and periodic boundary conditions determined by the magnetic translations.

Theorem 4.3 *We assume hypotheses (H3)-(H4), and let $I \subset \mathbb{R}$ be a bounded interval. There is a finite constant $C_W \equiv C_{B,u,I} > 0$, and a length scale $L_{B,I}$, so that for any subinterval $\Delta \subset I$ small enough, and for any $L \in \mathbb{N}_B$ with $L > L_{B,I}$, we have*

$$I\!E\{Tr(E_{\Lambda_L}(\Delta))\} \leq C_W s(|\Delta|)L^2,$$

where $s(\epsilon)$ is defined in (1.6).

Sketch of the Proof of Theorem 4.3.
1. We write Λ for Λ_L, where L is a permissible length as described above. Without loss of generality, we assume that I, and the subinterval $\Delta \subset I$ contains only the Landau level $E_n(B)$ and no other Landau level. Let $E_0 \in \Delta$ be the center of the interval. We write the decomposition in (2.4) using the unperturbed projector $\Pi_{n,L}$,

$$TrE_\Lambda(\Delta) = Tr\ E_\Lambda(\Delta)\Pi_{n,L} + TrE_\Lambda(\Delta)\Pi_{n,L}^{\perp}. \tag{4.3}$$

For the complementary term on the right in (4.3), we follow the argument in (2.5)–(2.14). We can take, for example, $M = 1$ in (2.7). We easily derive the analog of (2.9),

$$\begin{aligned} TrE_\Lambda(\Delta)\Pi_{n,L}^{\perp} &\leq K_n TrE_\Lambda(\Delta)V_\Lambda \frac{1}{(H_{\Lambda_L}^0(B)+1)^2} V_\Lambda E_\Lambda(\Delta) \\ &\leq K_n \sum_{i,j\in\tilde{\Lambda}} Tr\left[u_j E_\Lambda(\Delta)u_i \cdot \left(\chi_i \frac{1}{(H_{\Lambda_L}^0(B)+1)^2}\chi_j\right)\right], \end{aligned} \tag{4.4}$$

$$\tag{4.5}$$

where the constant K_n depends on the Landau level n and is expressible in the form of (2.11) with d_Δ there replaced by

$$d_n = \min(\text{ dist } (I, E_{n-1}(B)),\ \text{dist } (I, E_{n+1}(B))).$$

The operator $K_{ij} \equiv \chi_i(H_{\Lambda_L}^0(B)+1)^{-2}\chi_j$ is trace class (cf. [5]) and satisfies an exponential decay estimate analogous to (2.10). Completing the argument to (2.16), we obtain

$$I\!E\{TrE_\Lambda(\Delta)\Pi_{n,L}^{\perp}\} \leq K_n C_0 s(|\Delta|)L^2.$$

2. We now estimate the first term on the right in (2.3) using the unique continuation principle (4.2),

$$\begin{aligned} TrE_\Lambda(\Delta)\Pi_{n,L} &\leq \frac{1}{C(n,u)}\Big\{TrE_\Lambda(\Delta)\tilde{V}_\Lambda\Pi_{n,L} \\ &\qquad -TrE_\Lambda(\Delta)\Pi_{n,L}^{\perp}\tilde{V}_\Lambda\Pi_{n,L}\Big\}. \end{aligned} \tag{4.6}$$

We estimate the second term on the right in (4.6) as in (2.18)–(2.19), and obtain a bound similar to (2.20),

$$|TrE_\Lambda(\Delta)\Pi_{n,L}^\perp \tilde{V}_\Lambda \Pi_{n,L}| \leq \frac{1}{2\kappa_0} Tr\Pi_{n,L}^\perp E_\Lambda(\Delta) + \frac{\kappa_0 D_0^2}{2} TrE_\Lambda(\Delta)\Pi_{n,L}, \quad (4.7)$$

where we used the constant D_0 from (2.2). We now substitute (4.7) into the right of (4.6) and obtain the analog of (2.21),

$$\begin{aligned} \left(1 - \frac{\kappa_0 D_0^2}{2C(n,u)}\right) TrE_\Lambda(\Delta)\Pi_{n,L} \;\; &\leq \;\; \frac{1}{2\kappa_0 C(n,u)} TrE_\Lambda(\Delta)\Pi_{n,L}^\perp \\ &\quad + \frac{1}{C(n,u)} |TrE_\Lambda(\Delta)\tilde{V}_\Lambda \Pi_{n,L}|. \end{aligned} \quad (4.8)$$

We choose $\kappa_0 = C(n,u)/D_0^2$, and obtain from (4.8) an estimate for the left side of (4.6),

$$TrE_\Lambda(\Delta)\Pi_{n,L} \leq \frac{2}{C(n,u)} |TrE_\Lambda(\Delta)\tilde{V}_\Lambda \Pi_{n,L}| + \frac{D_0^2}{C(n,u)^2} TrE_\Lambda(\Delta)\Pi_{n,L}^\perp.$$

We follow the same method to estimate the first term on the right in (4.8) and obtain finally the analog of (2.23),

$$TrE_\Lambda(\Delta)\Pi_{n,L} \leq \frac{2D_0^2}{C(n,u)^2} TrE_\Lambda(\Delta)\Pi_{n,L}^\perp + \frac{4}{C(n,u)^2} Tr\Pi_{n,L}\tilde{V}_\Lambda E_\Lambda(\Delta)\tilde{V}_\Lambda \Pi_{n,L}.$$

3. We now estimate $Tr\Pi_{n,L}\tilde{V}_\Lambda E_\Lambda(\Delta)\tilde{V}_\Lambda\Pi_{n,L}$ as in the proof of Theorem 1.1. As in (2.24), we first write

$$\begin{aligned} Tr\Pi_{n,L}\tilde{V}_\Lambda E_\Lambda(\Delta)\tilde{V}_\Lambda\Pi_{n,L} &= TrE_\Lambda(\Delta)\tilde{V}_\Lambda\Pi_{n,L}\tilde{V}_\Lambda E_\Lambda(\Delta) \\ &\leq TrE_\Lambda(\Delta)\tilde{V}_\Lambda f_n(H^0_{\Lambda_L}(B))\tilde{V}_\Lambda\Pi_{n,L}, \end{aligned}$$

where $f_n \in C_0^\infty(\mathbb{R})$ is equal to one near $E_n(B)$. We expand the potential and obtain

$$Tr\Pi_{n,L}\tilde{V}_\Lambda E_\Lambda(\Delta)\tilde{V}_\Lambda\Pi_{n,L} \leq \sum_{i,j\in\tilde{\Lambda}} Tr\ u_j E_\Lambda(\Delta) u_i \cdot \chi_i f_n(H^0_{\Lambda_L}(B))\chi_j.$$

Following a similar analysis as from (2.26) to (2.30), we obtain

$$I\!\!E\{TrE_\Lambda(\Delta)\} \leq C_3(n,u)s(|\Delta|)L^2,$$

according to hypothesis (H4). □

6 Appendix: Trace-class Estimates

For the purposes of this appendix, we let $u \in L_0^\infty(\mathbb{R}^d)$ denote a compactly-supported function and write $u_j(x) = u(x-j)$, for $j \in \mathbb{Z}^d$. We note that the operator $K_{ij} \equiv u_i(H_0^\Lambda + M)^{-2}u_j$ (similar to the operator in (2.10)) is trace class for $d = 1, 2, 3$. For higher dimensions, $d > 3$, we proceed as follows. The operator $u_i(H_0^\Lambda + M)^{-1} \in \mathcal{I}_q$, where $\mathcal{I}_q$ is the q^{th}-von Neumann Schatten class, provided $q > d/2$ (cf. [27]). We state the essential properties in the following lemma.

Lemma 6.1 *Let $u \in L^\infty(\mathbb{R}^d)$ be a compactly-supported function centered about the origin and set $u_j(x) = u(x-j)$, for $j \in \tilde{\Lambda}$, so that u_j is a compactly-supported function centered about $j \in \tilde{\Lambda}$. We assume that $H_0^\Lambda + M$ is boundedly invertible for some $M > 0$, and for all Λ.*

1. *The bounded operator $K_{ij} \equiv u_i(H_0^\Lambda + M)^{-2}u_j$ is trace class if $u_iu_j = 0$. In this case, there are constants $c_0, C_0 > 0$, independent of Λ, and i, j, so that*
$$\|K_{ij}\|_1 = \|u_i(H_0^\Lambda + M)^{-2}u_j\|_1 \leq C_0 e^{-c_0\|i-j\|}.$$

2. *The operator $(H_0^\Lambda + M)^{-1}u_j \in \mathcal{I}_q$, for any $q > d/2$. Let $\mathcal{J}_i \equiv \{j \in \tilde{\Lambda} \mid u_iu_j \neq 0\}$, and define*
$$\tilde{K}_\Lambda \equiv \sum_{i\in\tilde{\Lambda}; j\in\mathcal{J}_i} u_iK_{ij}u_j.$$
Then, for any $m > 0$, any $\sigma_j > 0$, with $\sigma_0 = 1$, we can express the partial sum of the trace in (2.9) in the following form:
$$\left|\sum_{i\in\tilde{\Lambda}; j\in\mathcal{J}_i} Tr\ E_\Lambda(\Delta)\cdot u_iK_{ij}u_j\right| \leq \left(\sum_{j=1}^m \frac{\sigma_j}{2^j\sigma_1\cdots\sigma_{j-1}}\right) TrE_\Lambda(\Delta) + \left(\frac{1}{2^m\sigma_1\cdots\sigma_m}\right) Tr\ E_\Lambda(\Delta)\cdot\tilde{K}_\Lambda^{2^m}.$$
If $m + 2 > \log d/\log 2$, the operator $\tilde{K}_\Lambda^{2^m}$ is trace class and $\|\tilde{K}_\Lambda^{2^m}\|_1 \leq C(u, m, d)|\Lambda|$.

Proof.
1. *Disjoint Support, Off-Diagonal Terms.* We first consider separately the terms K_{ij} for which we have disjoint supports: $u_i u_j = 0$. Let $R_0 \equiv (H_0^\Lambda + M)^{-1}$ for notational convenience. Let χ, $\tilde{\chi}$, and $\tilde{\tilde{\chi}}$ be a smooth, compactly-supported function with values in $[0,1]$, and such that $\chi u = u$. We choose $\tilde{\chi}$ so that $\tilde{\chi}\chi = \chi$. We denote by $W(\chi)$ the first-order localized operator $W(\chi) \equiv [\chi, H_0]$, and we set $\chi_j(x) = \chi(x-j)$, similarly for $\tilde{\chi}$. If $u_i u_j = 0$, then we can choose χ and $\tilde{\chi}$ so that $\chi_j u_i = 0 = \tilde{\chi}_j u_i$. Finally, we take $\tilde{\tilde{\chi}}_j$ so that $\tilde{\tilde{\chi}}_j W(\tilde{\chi}_j) = W(\tilde{\chi}_j)$. In the disjoint support case, we have

$$
\begin{aligned}
u_i R_0^2 u_j &= u_i R_0^2 \chi_j u_j \\
&= u_i R_0^2 W(\chi_j) R_0 u_j + u_i R_0 \chi_j R_0 u_j \\
&= u_i R_0^2 W(\chi_j) R_0 u_j + u_i R_0 W(\chi_j) R_0^2 u_j \\
&= u_i R_0^2 W(\tilde{\chi}_j) R_0 W(\chi_j) R_0 u_j + u_i R_0 W(\tilde{\chi}_j) R_0^2 W(\chi_j) R_0 u_j \\
&\quad + u_i R_0 W(\tilde{\chi}_j) R_0 W(\chi_j) R_0^2 u_j.
\end{aligned} \tag{6.1}
$$

The operator $(H_0^\Lambda + M)^{-1} u_j \in \mathcal{I}_q$, for any $q > d/2$. If we suppose that $q = 3$, for example, then the Hölder inequality applied to the first term in (6.1) implies that

$$
\begin{aligned}
\|u_i R_0^2 W(\tilde{\chi}_j) R_0 W(\chi_j) R_0 u_j\|_1 &\leq \|u_i R_0\|_3 \, \|R_0 W(\tilde{\chi}_j) R_0 W(\chi_j)\|_3 \, \|R_0 u_j\|_3 \\
&\leq \|u_i R_0\|_3 \, \|R_0 \tilde{\tilde{\chi}}_j\|_3 \, \|R_0 u_j\|_3 \, \|W(\tilde{\chi}_j) R_0 W(\chi_j)\| \\
&< \infty,
\end{aligned} \tag{6.2}
$$

since the operator norm on the last line of (6.2) is bounded. It is clear that this extends the result to $d = 4, 5$. Iterating this scheme with finitely-many cut-off functions, and recalling that the operator $W(\chi_j)(H_0^\Lambda + M)^{-1} \in \mathcal{I}_q$, for any $q > d$, we see that $u_i R_0^2 u_j$ is trace class in any dimension provided $u_i u_j = 0$. The exponential decay in the trace norm can be proved using the Combes-Thomas method, cf. [2].
2. *Nondisjoint Support Terms.* Let $\|A\|_2$ denote the Hilbert-Schmidt norm of an operator A. For $i \in \tilde{\Lambda}$, we let $\mathcal{J}_i \equiv \{ j \in \tilde{\Lambda} \mid u_i u_j \neq 0 \}$, and define

$$
\tilde{K}_\Lambda \equiv \sum_{i \in \tilde{\Lambda}; j \in \mathcal{J}_i} u_i K_{ij} u_j .
$$

Note that $|\mathcal{J}_i|$ is finite, independent of i, depends only on supp u, and so is independent of $|\Lambda|$. Then we can express the sum of the nondisjoint support

terms occurring in (2.9) in the following form:

$$\begin{aligned}
\left| \sum_{i \in \tilde{\Lambda}; j \in \mathcal{J}_i} Tr\ u_j E_\Lambda(\Delta) u_i \cdot K_{ij} \right| &= |Tr E_\Lambda(\Delta) \tilde{K}_\Lambda| \\
&\leq \|E_\Lambda(\Delta)\|_2 \|E_\Lambda(\Delta) \tilde{K}_\Lambda\|_2 \\
&\leq \frac{\sigma_1}{2} Tr E_\Lambda(\Delta) + \frac{1}{2\sigma_1} Tr E_\Lambda(\Delta) \tilde{K}_\Lambda^2,
\end{aligned}$$

for any $\sigma_1 > 0$. We iterate this expression m times and obtain

$$\begin{aligned}
|Tr E_\Lambda(\Delta) \tilde{K}_\Lambda| &\leq \left(\sum_{j=1}^{m} \frac{\sigma_j}{2^j \sigma_1 \cdots \sigma_{j-1}} \right) Tr E_\Lambda(\Delta) \\
&\quad + \left(\frac{1}{2^m \sigma_1 \cdots \sigma_m} \right) Tr\ E_\Lambda(\Delta) \cdot \tilde{K}_\Lambda^{2^m}
\end{aligned}$$

where $\sigma_0 \equiv 1$. To describe the operator $\tilde{K}_\Lambda^n$, we define an index set $\mathcal{J}_{j_k} \equiv \{m \in \tilde{\Lambda} \mid u_m u_{j_k} \neq 0\}$. We can then write

$$\tilde{K}_\Lambda^n = \sum_{i \in \tilde{\Lambda}; j_k \in \mathcal{J}_{j_{k-1}}, k=1,\ldots,n; j_0 = i} u_i^2 R_0^2 u_{j_1}^2 u_{j_2}^2 R_0^2 u_{j_3}^2 u_{j_4}^2 R_0^2 \cdots u_{j_{n-1}}^2 R_0^2 u_{j_n}^2.$$

Since $u_i R_0^2 u_j \in \mathcal{I}_q$, for any $q > d/4$, Hölder's inequality implies that $\tilde{K}_\Lambda^n \in \mathcal{I}_1$ if $n > d/4$. It is clear then for $m + 2 > \log d / \log 2$, the operator $\tilde{K}_\Lambda^{2^m} \in \mathcal{I}_1$. Finally, we easily estimate the trace norm:

$$\|\tilde{K}_\Lambda^{2^m}\|_1 \leq C(u, m, d) |\Lambda|,$$

for a constant $0 < C(u, m, d) < \infty$ independent of $|\Lambda|$. $\square$

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